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Matrix pencils and existence conditions for quadratic programming with a sign-indefinite quadratic equality constraint

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Abstract We consider minimization of a quadratic objective function subject to a sign-indefinite quadratic equality constraint. We derive necessary and sufficient conditions for the existence of solutions to the constrained minimization problem. These conditions involve a generalized eigenvalue of the matrix pencil consisting of a symmetric positive-semidefinite matrix and a symmetric indefinite matrix. A complete characterization of the solution set to the constrained minimization problem in terms of the eigenspace of the matrix pencil is provided.

Keywords Matrix pencil · Quadratic programming · Existence theory

1 Introduction

We consider the following quadratically constrained quadratic programming (QCQP) problem. Let $M \in \mathbb{R}^{n\times n}$ be a (symmetric) positive-semidefinite matrix, let $\theta \in \mathbb{R}^n$, let $N \in \mathbb{R}^{n\times n}$ be a symmetric matrix with at least one positive eigenvalue, let γ be a positive number, and consider the problem

$$\min \theta^{\mathrm{T}} M\theta \tag{1}$$

subject to

$$\theta^{\mathrm{T}} N\theta = \gamma. \tag{2}$$

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Note that, the objective function (1) is convex, while the equality constraint (2) is possibly sign indefinite. The specific motivation for studying this problem is discussed below.

Variations of (1), (2) have been extensively considered in the literature. Quadratic programming problems with a quadratic objective function and a quadratic inequality constraint (both of which may contain linear terms) are considered in [1,3,5,11–13,18,19,21,23,27]. These works apply duality concepts and variational methods to characterize and compute global minimizers.

Quadratic programming problems are closely associated with least squares optimization subject to convex quadratic constraints [8,10,20,32]. In particular, in least squares optimization problems with noise residuals that are uncorrelated with the coefficient matrix and with each other, standard least squares solutions are known to be the best linear unbiased estimator (BLUE) [7, Chapter 6]. Along the same lines, the specific motivation for the present paper is time series model identification under noisy measurements [17,26]. For this class of problems, the coefficient matrix is comprised of measurements of the input and output signals, and thus is correlated with the noise residual. This situation, which is not addressed by BLUE theory, can lead to biased estimators [26, pp. 66, 187].

The present paper is motivated by the observation that, the true parameters of a time-series model are given by the solution of a least squares optimization problem involving a homogeneous quadratic positive-semidefinite objective function and a homogeneous sign-indefinite quadratic equality constraint. Relevant details as well as references to the system identification literature on unbiased least squares are given in [15,16,22,25,30,31,33].

One approach to guaranteeing that QCQP has a solution is to observe that the constraint set is the intersection of the set

$$\mathbb{S} \triangleq \left\{ \begin{bmatrix} \theta^{\mathrm{T}} M \theta \\ \theta^{\mathrm{T}} N \theta \end{bmatrix} : \theta \in \mathbb{R}^n \right\} \subseteq \mathbb{R}^2 \tag{3}$$

and a horizontal line passing though γ on the vertical axis in $\mathbb{R}^2$. Since M is positive semidefinite, it follows that this intersection is bounded from the left by the vertical axis. Thus QCQP has a solution if $\mathbb{S}$ is closed. In [9], it is shown that, $\mathbb{S}$ is convex (see also [24]). Furthermore, [9, Theorem 2] states that if, for all $\theta \in \mathbb{R}^n \neq 0$,

$$\theta^{\mathrm{T}} N \theta = 0 \quad \text{implies } \theta^{\mathrm{T}} M \theta \neq 0, \tag{4}$$

then $\mathbb{S}$ is closed. However, as illustrated by Example 8 in Table 1, the condition given in [9] is sufficient, but not necessary. Moreover, the results in [9] are not useful when (4) does not hold.

In the present paper we adopt a matrix pencil approach to obtain necessary and sufficient conditions for the existence of solutions to QCQP. In particular, we characterize the solution set of QCQP in terms of a nonnegative generalized eigenvalue of the matrix pencil formed from the objective matrix and the constraint matrix. The principal contribution of the paper is a complete characterization of the existence of solutions as well as the solution set in terms of the properties of the matrix pencil.

A matrix pencil approach to a related constrained quadratic programming problem is given in [27], where the objective function includes a linear term (see problem (P) in Sect. 2 of [27]). The constraint in [27] is given by a sign-indefinite homogeneous quadratic form with upper and lower bounds. This problem specializes to QCQP when the linear term in the objective function is zero, the objective function is positive-semidefinite, and the upper and lower bounds are equal (that is, $\alpha = \beta$ in the notation of [27], see problem (P_T) in Sect. 3 of [27]). However, unlike [27] we do not assume that the constraint matrix is nonsingular. While Theorem 2.1 of [27] gives sufficient (and, under a constraint qualification, necessary)

Table 1 QCQP examples. In all examples $\lambda_{\max}(N) > 0$

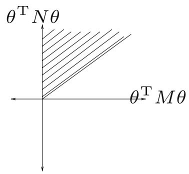
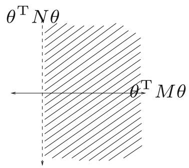
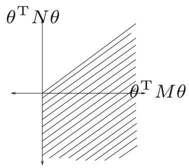
Example	M	N	(M, N) regular?	$\alpha_{\max}$	Generalized eigenvalues	# QCQP solutions	min $\mathcal{J}$	$\mathbb{S}$	Is $\mathbb{S}$ closed?
1	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	Yes	0	0, 1	2	0		Yes
2	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$	Yes	0	0, 0	0	Does not exist		No
3	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Yes	1	0, 1	2	γ		Yes

Table 1 continued

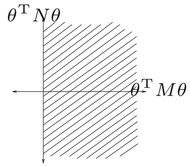
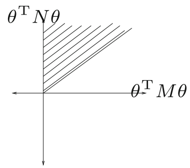
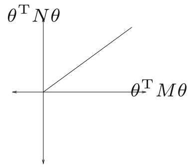
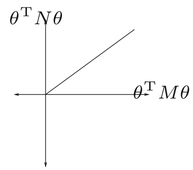
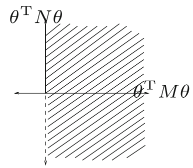
Example	M	N	(M, N) regular?	$\alpha_{\max}$	Generalized eigenvalues	# QCQP solutions	$\min \mathcal{J}$	$\mathbb{S}$	Is $\mathbb{S}$ closed?
4	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	Yes	0	0, 0, 1	2	0		Yes
5	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	No	0	All $\lambda \in \mathbb{C}$	∞	0		Yes
6	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	No	1	All $\lambda \in \mathbb{C}$	∞	γ		Yes

Table 1 continued

Example	M	N	(M, N) regular?	$\alpha_{\max}$	Generalized eigenvalues	# QCQP solutions	$\min \mathcal{J}$	$\mathbb{S}$	Is $\mathbb{S}$ closed?
7	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	Yes	1	1, 1	∞	γ		Yes
8	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	Yes	0	0, 0, 0	∞	0		No

conditions for a feasible point to be a global minimizer, existence theory is not addressed in [27].

Matrix pencils are also used in [21] to analyze existence of solutions in a quadratic programming problem with quadratic, but not necessarily homogeneous, objective and constraint functions. As in [27], the results of [21] are derived by means of Lagrange multiplier theory. Accordingly, a constraint qualification is invoked in [21], specifically, that the constraint function is indefinite. The approach of the present paper avoids the use of Lagrange multipliers, and thus no constraint qualification is needed.

The analysis that supports these results is self contained in the sense that, we do not rely on results on matrix pencils from the literature. In particular, the pencil of interest is a symmetric pencil involving a positive-semidefinite matrix and a symmetric matrix having at least one positive eigenvalue. Since the literature on matrix pencils [2,4,6,14,28,29] does not address this specific problem, we provide supporting proofs for all of the results that are needed.

Notation $0_{n\times m}$ is the $n \times m$ zero matrix, I_n is the $n \times n$ identity matrix, $\mathbb{R}^n(\mathbb{C}^n)$ is the set of real (complex) $n \times 1$ column vectors, and $\mathbb{R}^{n\times m}(\mathbb{C}^{n\times m})$ is the set of real (complex) $n \times m$ matrices. For $A \in \mathbb{C}^{n\times m}$, A^* is the complex conjugate transpose of A. For $A \in \mathbb{R}^{n\times m}$, rank A is the rank of A, $\mathcal{R}(A)$ is the range of A, $\mathcal{N}(A)$ is the null space of A, and def A is the defect (nullity) of A. For symmetric $A \in \mathbb{R}^{n\times n}$, $\lambda_{\max}(A)$ is the largest eigenvalue of A, $\lambda_{\min}(A)$ is the smallest eigenvalue of A, $A > 0$ means that A is positive definite, and $A \geq 0$ means that A is positive semidefinite. Finally, diag $(a_1, \ldots, a_n)$ is the diagonal matrix with diagonal entries $a_1, \ldots, a_n$.

2 Quadratic programming problem

We consider the *quadratic objective function*

$$\mathcal{J}(\theta) = \theta^{\mathrm{T}} M\theta, \tag{5}$$

where $\theta \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$ is positive-semidefinite. Let $\gamma > 0$, let $N \in \mathbb{R}^{n\times n}$ be symmetric, and define the *parameter constraint set* $\mathcal{D}_\gamma(N)$ by

$$\mathcal{D}_\gamma(N) \triangleq \{\theta \in \mathbb{R}^n : \theta^{\mathrm{T}} N\theta = \gamma\}. \tag{6}$$

The *quadratically constrained quadratic programming* (QCQP) *problem* is then given by

$$\min_{\theta\in\mathcal{D}_\gamma(N)} \mathcal{J}(\theta). \tag{7}$$

We do not require that N be positive-semidefinite, however, a negative definite N results in an empty $\mathcal{D}_\gamma(N)$ and is not useful. Therefore, we will later make the assumption that, N is not negative definite.

The following result concerns properties of $\mathcal{D}_\gamma(N)$. By convention, the empty set is convex.

Proposition 2.1 *The set $\mathcal{D}_\gamma(N)$ has the following properties:*

(i) $\mathcal{D}_\gamma(N)$ *is closed.*
(ii) $\mathcal{D}_\gamma(N)$ *is symmetric, that is,* $\theta \in \mathcal{D}_\gamma(N)$ *if and only if* $-\theta \in \mathcal{D}_\gamma(N)$.
(iii) The following statements are equivalent

(a) $\mathcal{D}_\gamma(N) \neq \varnothing$.
(b) $\mathcal{D}_\gamma(N)$ *is not convex.*
(c) $\lambda_{\max}(N) > 0$.

(iv) $\mathcal{D}_\gamma(N) \neq \varnothing$ *and compact if and only if* N *is positive definite.*
(v) *If* $\mathcal{D}_\gamma(N) \neq \varnothing$ *and* $n > 1$, *then* $\mathcal{D}_\gamma(N)$ *has uncountably many elements.*

Next, define the *solution set* $\mathcal{W}_\gamma(N)$ by

$$\mathcal{W}_\gamma(N) \triangleq \{\theta \in \mathcal{D}_\gamma(N) : \mathcal{J}(\theta) = \min_{\theta' \in \mathcal{D}_\gamma(N)} \mathcal{J}(\theta')\}. \tag{8}$$

Proposition 2.2 *The solution set* $\mathcal{W}_\gamma(N)$ *has the following properties:*

(i) $\mathcal{W}_\gamma(N)$ *is closed.*
(ii) $\mathcal{W}_\gamma(N)$ *is symmetric.*
(iii) *If* $\mathcal{W}_\gamma(N) \neq \varnothing$ *then the QCQP problem (7) has at least two solutions. In particular,* $\theta \in \mathbb{R}^n$ *solves the QCQP problem (7) if and only if* $-\theta$ *does.*

3 Matrix pencil and generalized eigenvalues

Let $A,\ B \in \mathbb{R}^{p\times p}$. Then the *matrix pencil* $P_{A,B}(s)$ is defined by,

$$P_{A,B}(s) \triangleq A - sB. \tag{9}$$

Furthermore, define the *characteristic polynomial* $\chi_{A,B}(s)$ by

$$\chi_{A,B}(s) \triangleq \det(A - sB). \tag{10}$$

The pair $(A,\ B)$ is *regular* if $\chi_{A,B}(s)$ is not the zero polynomial. The roots of $\chi_{A,B}(s)$ are the *generalized eigenvalues* of $(A,\ B)$.

Lemma 3.1 $\mathcal{N}(M) \cap \mathcal{N}(N) = \mathcal{N}(M) \cap \mathcal{N}(M-N) = \mathcal{N}\left(\begin{bmatrix} M \\ N \end{bmatrix}\right)$.

Proof

$$\begin{aligned}\mathcal{N}(M) \cap \mathcal{N}(M-N) &= \mathcal{N}\left(\begin{bmatrix} M \\ M-N \end{bmatrix}\right) \\ &= \mathcal{N}\left(\begin{bmatrix} I_n & 0_{n\times n} \\ I_n & -I_n \end{bmatrix}\begin{bmatrix} M \\ N \end{bmatrix}\right) \\ &= \mathcal{N}\left(\begin{bmatrix} M \\ N \end{bmatrix}\right) \\ &= \mathcal{N}(M) \cap \mathcal{N}(N).\end{aligned}$$

□

Proposition 3.2 *The following statements are equivalent:*

(i) (M, N) *is regular.*
(ii) *There exists* $\alpha \in \mathbb{R}$ *such that* $M + \alpha N$ *is nonsingular.*
(iii) $\mathcal{N}(M) \cap \mathcal{N}(N) = \{0\}$.
(iv) $\mathcal{N}\left(\begin{bmatrix} M \\ N \end{bmatrix}\right) = \{0\}$.

(v) $\mathcal{N}(M) \cap \mathcal{N}(M-N) = \{0\}$.
(vi) *All generalized eigenvalues of* (M, N) *are real.*
If, in addition, $N \geq 0$, then the following statement is equivalent to (i)–(vi):
(vii) *There exists* $\beta \in \mathbb{R}$ *such that* $\beta N < M$.

Proof The results (i) implies (ii) and (ii) implies (iii) are immediate. Next, Lemma 3.1 implies that (iii), (iv), and (v) are equivalent. Next, to prove (iii) implies (vi), let $\lambda \in \mathbb{C}$ be a generalized eigenvalue of (M, N). Since $\lambda = 0$ is real, suppose $\lambda \neq 0$. Since $\det(M - \lambda N) = 0$, let nonzero $\theta \in \mathbb{C}^n$ satisfy $(M - \lambda N)\theta = 0$ and thus, it follows that, $\theta^* M\theta = \lambda\theta^* N\theta$. Furthermore, note that $\theta^* M\theta$ and $\theta^* N\theta$ are real. Now, suppose $\theta \in \mathcal{N}(M)$. Then it follows from $(M - \lambda N)\theta = 0$ that $\theta \in \mathcal{N}(N)$, which contradicts $\mathcal{N}(M) \cap \mathcal{N}(N) = \{0\}$. Hence $\theta \notin \mathcal{N}(M)$ and thus, $\theta^* M\theta > 0$ and consequently $\theta^* N\theta \neq 0$. Hence, it follows that $\lambda = \theta^* M\theta/\theta^* N\theta$, and thus, λ is real. Hence all generalized eigenvalues of (M, N) are real. Next, to prove (vi) implies (i), let $\lambda \in \mathbb{C}\backslash\mathbb{R}$ so that λ is not a generalized eigenvalue of (M, N). Consequently, $\chi_{M,N}(s)$ is not the zero polynomial, and thus (M, N) is regular.

Next, to prove $(i) - (vi)$ imply (vii), let $\theta \in \mathbb{R}^n$ be nonzero and note that, $\mathcal{N}(M) \cap \mathcal{N}(N) = \{0\}$ implies that either $M\theta \neq 0$ or $N\theta \neq 0$. Hence, either $\theta^{\mathrm{T}} M\theta > 0$ or $\theta^T N\theta > 0$. Thus, $\theta^T(M + N)\theta > 0$, which implies $M + N > 0$ and hence $-N < M$.

Finally, to prove (vii) implies $(i) - (vi)$, let $\beta \in \mathbb{R}$ be such that $\beta N < M$, so that $\beta\theta^T N\theta < \theta^T M\theta$ for all nonzero $\theta \in \mathbb{R}^n$. Next, suppose $\hat{\theta} \in \mathcal{N}(M) \cap \mathcal{N}(N)$ is nonzero. Hence $M\hat{\theta} = 0$ and $N\hat{\theta} = 0$. Consequently, $\hat{\theta}^T N\hat{\theta} = 0$ and $\hat{\theta}^{\mathrm{T}} M\hat{\theta} = 0$, which contradicts $\beta\hat{\theta}^T N\hat{\theta} < \hat{\theta}^T M\hat{\theta}$. Thus, $\mathcal{N}(M) \cap \mathcal{N}(N) = \{0\}$. □

Corollary 3.3 *Assume* $\theta^{\mathrm{T}} M\theta > 0$ *for all nonzero* $\theta \in \mathbb{R}^n$ *satisfying* $\theta^{\mathrm{T}} N\theta = 0$. *Then* (M, N) *is regular.*

Proof Since $\mathcal{N}(N) \subseteq \mathcal{Q} \triangleq \{\theta : \theta^{\mathrm{T}} N\theta = 0\}$, and by assumption $\theta^{\mathrm{T}} M\theta > 0$ for all nonzero $\theta \in \mathcal{Q}$, it follows that $\mathcal{N}(M) \cap \mathcal{N}(N) = \{0\}$. Thus (M, N) is regular. □

The converse of Corollary 3.3 is not true. In Example 4 in Table 1, (M, N) is regular, but $\theta = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^{\mathrm{T}}$ satisfies $\theta^{\mathrm{T}} M\theta = \theta^{\mathrm{T}} N\theta = 0$.

4 Properties of $\alpha_{\max}$

In this section, we discuss the properties of $\alpha_{\max}$, which is defined in terms of the set

$$\mathcal{S} \triangleq \{\alpha \geq 0 : \alpha N \leq M\}. \tag{11}$$

Next, define $\alpha_{\max} \triangleq \max \mathcal{S}$. Proposition 4.1 shows that, $\mathcal{S}$ is compact and thus, $\alpha_{\max}$ exists.

For the remainder of the paper, we assume that $\lambda_{\max}(N) > 0$.

Proposition 4.1 *The set* $\mathcal{S}$ *has the following properties:*

(i) $0 \in \mathcal{S} \neq \varnothing$.
(ii) $\mathcal{S}$ *is compact.*
(iii) $\mathcal{S}$ *is connected.*

Proof To prove (i), note that $M \geq 0$ implies that $0 \in \mathcal{S}$. To prove (ii) note that $f : [0, \infty) \to \mathbb{R}$ defined by $f(\alpha) \triangleq \lambda_{\min}(M - \alpha N)$ is continuous. Hence $\mathcal{S} = f^{-1}([0, \infty))$ is closed. To prove that $\mathcal{S}$ is bounded, let the orthogonal matrix $U \in \mathbb{R}^{n\times n}$ be such that

$\hat{N} \triangleq UNU^{\mathrm{T}}$ is diagonal and $\hat{N}_{1,1} = \lambda_{\max}(N)$, where $\hat{N}_{1,1}$ is the $(1,\ 1)$ entry of $\hat{N}$. Then $\alpha \in \mathcal{S}$ implies that $\alpha\hat{N} \le \hat{M} \triangleq UMU^{\mathrm{T}}$. Hence, $\alpha \in \mathcal{S}$ satisfies $\alpha\lambda_{\max}(N) \le \hat{M}_{1,1}$, which implies $\alpha \le \hat{M}_{1,1}/\lambda_{\max}(N)$. To prove (iii), let $\alpha \in \mathcal{S}$ be positive and let $\hat{\alpha} \in (0,\ \alpha)$. Define $\beta \triangleq \hat{\alpha}/\alpha$ and note that $0 < \beta < 1$. Then it follows from $\alpha N \le M$ that $\hat{\alpha}N \le \beta M \le M$, and thus $\hat{\alpha}N \le M$. Hence $\hat{\alpha} \in \mathcal{S}$, and thus $\mathcal{S}$ is connected. □

It follows from Proposition 4.1 that $\mathcal{S} = [\,0, \alpha_{\max}\,]$.

Proposition 4.2 *If* $N \le M$, *then* $\alpha_{\max} \ge 1$.

The following result shows that $M - \alpha_{\max}N$ has a nontrivial null space.

Proposition 4.3 $\alpha_{\max}$ *is a generalized eigenvalue of* $(M,\ N)$.

Proof To prove $\det(M - \alpha_{\max}N) = 0$, let $\alpha \ge 0$. Then $\lambda_{\min}(M - \alpha N) \ge 0$ for all $\alpha \le \alpha_{\max}$, whereas $\lambda_{\min}(M - \alpha N) < 0$ for all $\alpha > \alpha_{\max}$. Taking limits, it follows that $\lim_{\alpha\uparrow\alpha_{\max}} \lambda_{\min}(M - \alpha N) \ge 0$ and $\lim_{\alpha\downarrow\alpha_{\max}} \lambda_{\min}(M - \alpha N) \le 0$. Since $\lambda_{\min}(M - \alpha N)$ is a continuous function of α,

$$0 \le \lim_{\alpha\uparrow\alpha_{\max}} \lambda_{\min}(M - \alpha N) = \lim_{\alpha\to\alpha_{\max}} \lambda_{\min}(M - \alpha N) = \lim_{\alpha\downarrow\alpha_{\max}} \lambda_{\min}(M - \alpha N) \le 0.$$

Hence $\lambda_{\min}(M - \alpha_{\max}N) = 0$, and thus $\det(M - \alpha_{\max}N) = 0$. □

Proposition 4.4

$$\lambda_{\min}(M)/\lambda_{\max}(N) \le \alpha_{\max} \le \lambda_{\max}(M)/\lambda_{\max}(N). \tag{12}$$

Proof The upper bound for $\alpha_{\max}$ follows immediately from $M - \alpha_{\max}N \ge 0$. To prove the lower bound, it follows that, for all nonzero $\theta \in \mathbb{R}^n$, $\theta^{\mathrm{T}}M\theta \ge \lambda_{\min}(M)\theta^{\mathrm{T}}\theta$ and $\theta^{\mathrm{T}}N\theta \le \lambda_{\max}(N)\theta^{\mathrm{T}}\theta$. Now let $\theta \in \mathcal{N}(M - \alpha_{\max}N)$ be nonzero. Then, $0 = \theta^{\mathrm{T}}(M - \alpha_{\max}N)\theta = \theta^{\mathrm{T}}M\theta - \alpha_{\max}\theta^{\mathrm{T}}N\theta \ge \lambda_{\min}(M)\theta^{\mathrm{T}}\theta - \alpha_{\max}\lambda_{\max}(N)\theta^{\mathrm{T}}\theta = (\lambda_{\min}(M) - \alpha_{\max}\lambda_{\max}(N))\theta^{\mathrm{T}}\theta$. Since $\theta^{\mathrm{T}}\theta > 0$, it follows that $\lambda_{\min}(M) - \alpha_{\max}\lambda_{\max}(N) \le 0$. □

The following basic result is used frequently in the subsequent development without comment.

Fact 4.5 *Let* $A \in \mathbb{R}^{n\times n}$. *Then* $\mathcal{N}(A) \subseteq \{\theta \in \mathbb{R}^n : \theta^{\mathrm{T}}A\theta = 0\}$. *If, in addition,* $A \ge 0$, *then* $\mathcal{N}(A) = \{\theta \in \mathbb{R}^n : \theta^{\mathrm{T}}A\theta = 0\}$.

Proposition 4.6 *Assume* $\alpha_{\max} > 0$, *let* $p \triangleq \mathit{dim}(\mathcal{N}(M) \cap \mathcal{N}(N))$, *and let* $\alpha_1, \alpha_2 \in (0,\ \alpha_{\max})$. *Then*

$$p = \mathit{def}(M - \alpha_1 N) = \mathit{def}(M - \alpha_2 N) < \mathit{def}(M - \alpha_{\max}N). \tag{13}$$

In particular, if $(M,\ N)$ *is regular, then*

$$0 = \mathit{def}(M - \alpha_1 N) = \mathit{def}(M - \alpha_2 N) < \mathit{def}(M - \alpha_{\max}N). \tag{14}$$

Proof First, we consider the case in which $(M,\ N)$ is regular. By Proposition 4.3, $\mathrm{def}(M - \alpha_{\max}N) > 0$. Next, suppose $\mathrm{def}(M - \alpha_1 N) > 0$, and let $\theta_1 \in \mathcal{N}(M - \alpha_1 N)$ be nonzero. It follows that $\theta_1^{\mathrm{T}}M\theta_1 = \alpha_1\theta_1^{\mathrm{T}}N\theta_1$. Furthermore, since $\theta_1^{\mathrm{T}}(M - \alpha_{\max}N)\theta_1 \ge 0$, it follows that $\theta_1^{T}M\theta_1 \ge \alpha_{\max}\theta_1^{\mathrm{T}}N\theta_1$. Consequently, $\alpha_1\theta_1^{\mathrm{T}}N\theta_1 \ge \alpha_{\max}\theta_1^{\mathrm{T}}N\theta_1$, and thus $(\alpha_1 - \alpha_{\max})\theta_1^{\mathrm{T}}N\theta_1 \ge 0$. Since $\alpha_1 < \alpha_{\max}$, it follows that $\theta_1^{\mathrm{T}}N\theta_1 \le 0$. However, $\theta_1^{\mathrm{T}}N\theta_1 =$

$\theta_1^{\mathrm{T}} M\theta_1/\alpha_1 \geq 0$. Thus, $\theta_1^{\mathrm{T}} N\theta_1 = 0$, which implies that $\theta_1^{\mathrm{T}} M\theta_1 = 0$ and thus $\theta_1 \in \mathcal{N}(M)$. Since $\alpha_1 > 0$ and $\theta_1 \in \mathcal{N}(M - \alpha_1 N) \cap \mathcal{N}(M)$, it follows that $\theta_1 \in \mathcal{N}(N)$, which by Proposition 3.2 contradicts the assumption that (M, N) is regular. Thus, (14) holds.

Next, suppose (M, N) is not regular so that, $p \triangleq \dim(\mathcal{N}(M) \cap \mathcal{N}(N)) > 0$. Let the nonsingular matrix $S \in \mathbb{R}^{n\times n}$ be such that $\hat{N} \triangleq SNS^{\mathrm{T}} = \mathrm{diag}(I_r, -I_s, 0_{(n-r-s)\times(n-r-s)})$, where r and s are the numbers of positive and negative eigenvalues of N, respectively. Next, define $\hat{M} \triangleq SMS^{\mathrm{T}}$ and note that $p = \dim(\mathcal{N}(\hat{M}) \cap \mathcal{N}(\hat{N}))$. Since the columns of $\begin{bmatrix} 0_{(r+s)\times(n-r-s)} \\ I_{n-r-s} \end{bmatrix}$ comprise a basis for $\mathcal{N}\left(\hat{N}\right)$, it follows that the trailing $(n-r-s) \times (n-r-s)$ submatrix of $\hat{M}$ has defect p. Now, applying a basis transformation to the last $n-r-s$ columns and rows of $\hat{M}$ and $\hat{N}$ if necessary (and without renaming $\hat{M}$ and $\hat{N}$) such that the last p rows and columns of $\hat{M}$ and $\hat{N}$ are zero, we partition $\hat{M}$ and $\hat{N}$ as

$$\hat{M} = \begin{bmatrix} \hat{M}_1 & 0_{(n-p)\times p} \\ 0_{p\times(n-p)} & 0_{p\times p} \end{bmatrix}, \quad \hat{N} = \begin{bmatrix} \hat{N}_1 & 0_{(n-p)\times p} \\ 0_{p\times(n-p)} & 0_{p\times p} \end{bmatrix},$$

where $\hat{M}_1, \ \hat{N}_1 \in \mathbb{R}^{(n-p)\times(n-p)}$. Since $p = \dim(\mathcal{N}(\hat{M}) \cap \mathcal{N}(\hat{N})) = \dim\Big((\mathcal{N}(\hat{M}_1) \oplus \mathbb{R}^p) \cap (\mathcal{N}(\hat{N}_1) \oplus \mathbb{R}^p)\Big) = \dim(\mathcal{N}(\hat{M}_1) \cap \mathcal{N}(\hat{N}_1)) + p$, it follows that, $\dim(\mathcal{N}(\hat{M}_1) \cap \mathcal{N}(\hat{N}_1)) = 0$ and thus $(\hat{M}_1, \ \hat{N}_1)$ is regular. Now, using (14), it follows that, $p = p + \mathrm{def}(\hat{M}_1 - \alpha_1 \hat{N}_1) = p + \mathrm{def}(\hat{M}_1 - \alpha_2 \hat{N}_1) = \mathrm{def}(\hat{M} - \alpha_1 \hat{N}) = \mathrm{def}(\hat{M} - \alpha_2 \hat{N}) < \mathrm{def}(M - \alpha_{\max} N)$ and hence (13) holds. □

Corollary 4.7 *Assume $\alpha_{\max} > 0$ and (M, N) is regular. Then $\alpha_{\max}$ is the smallest positive generalized eigenvalue of (M, N).*

The following result shows that, if M is not positive definite then $\alpha_{\max} = 0$ is possible, in which case $\mathcal{S} = \{0\}$.

Proposition 4.8 *The following statements hold:*

- *(i)* *If $\theta^{\mathrm{T}} N\theta < 0$ for all nonzero $\theta \in \mathcal{N}(M)$, then $\alpha_{\max} > 0$.*
- *(ii)* *If there exists $\theta \in \mathcal{N}(M)$ such that $\theta^{\mathrm{T}} N\theta > 0$, then $\alpha_{\max} = 0$.*
- *(iii)* *If (M, N) is regular and there exists nonzero $\theta \in \mathbb{R}^n$ such that $\theta^{\mathrm{T}} M\theta = \theta^{\mathrm{T}} N\theta = 0$, then $\alpha_{\max} = 0$.*

Proof To prove (i), let $\theta \in \mathbb{R}^n$. If $\theta \in \mathcal{N}(M)$ is nonzero, then by assumption $-\alpha\theta^{\mathrm{T}} N\theta > 0$ for all $\alpha > 0$ and thus, $\theta^{\mathrm{T}}(M - \alpha N)\theta > 0$ for all $\alpha > 0$. Next, suppose that $\theta \notin \mathcal{N}(M)$. Then $\theta^{\mathrm{T}} M\theta > 0$ and thus there exists $\alpha > 0$ such that $\theta^{\mathrm{T}}(M - \alpha N)\theta = \theta^{\mathrm{T}} M\theta - \alpha\theta^{\mathrm{T}} N\theta > 0$. Hence, for all $\theta \in \mathbb{R}^n$, it follows that there exists $\alpha > 0$ such that $\theta^{\mathrm{T}}(M - \alpha N)\theta > 0$. Thus $\alpha_{\max} > 0$. To prove (ii), suppose that $\alpha_{\max} > 0$. Then it follows that $\theta^{\mathrm{T}}(M - \alpha_{\max} N)\theta = -\alpha_{\max}\theta^{\mathrm{T}} N\theta < 0$, which contradicts $M - \alpha_{\max} N \geq 0$. Thus $\alpha_{\max} = 0$.

Finally, to prove (iii), let $\theta_1 \in \mathbb{R}^n$ be nonzero such that $\theta_1^{\mathrm{T}} M\theta_1 = \theta_1^{\mathrm{T}} N\theta_1 = 0$. Since $M \geq 0$ it follows that $\theta_1 \in \mathcal{N}(M)$. Furthermore, since (M, N) is regular, $\theta_1 \notin \mathcal{N}(N)$. Now, suppose $\alpha_{\max} > 0$. Hence $\mathcal{N}(M) \cap \mathcal{N}(\alpha_{\max} N) = \{0\}$ and thus from Lemma 3.1 it follows that $\mathcal{N}(M) \cap \mathcal{N}(M - \alpha_{\max} N) = \{0\}$. Next, note that $\theta_1^{\mathrm{T}}(M - \alpha_{\max} N)\theta_1 = 0$. Since, by definition, $M - \alpha_{\max} N \geq 0$, it follows that $\theta_1 \in \mathcal{N}(M - \alpha_{\max} N)$, which contradicts $\mathcal{N}(M) \cap \mathcal{N}(M - \alpha_{\max} N) = \{0\}$ and hence, $\alpha_{\max} = 0$. □

Corollary 4.9 *If $M > 0$ then $\alpha_{\max} > 0$.*

Corollary 4.10 *If* $\alpha_{\max} = 0$ *then there exists nonzero* $\theta \in \mathcal{N}(M)$ *such that* $\theta^{\mathrm{T}} N\theta \geq 0$.

Corollary 4.11 *If* $N > 0$ *and* M *is singular then* $\alpha_{\max} = 0$.

To illustrate Corollary 4.10, consider Example 1 in Table 1, where $\theta = \begin{bmatrix} 0 & 1 \end{bmatrix}^{\mathrm{T}}$ satisfies $\theta \in \mathcal{N}(M)$ and $\theta^{\mathrm{T}} N\theta > 0$. Furthermore, in Example 2, $\theta = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$ satisfies $\theta \in \mathcal{N}(M)$ and $\theta^{\mathrm{T}} N\theta = 0$.

Example 3 in Table 1 shows that $\alpha_{\max}$ is not necessarily the smallest nonnegative generalized eigenvalue of a regular pencil (M, N).

5 Existence of solutions

In this section we determine conditions under which $\mathcal{W}_\gamma(N) \neq \varnothing$, that is, the QCQP problem (7) has a solution.

The following result provides a lower bound for $\mathcal{J}(\theta)$.

Proposition 5.1 *For all* $\theta \in \mathcal{D}_\gamma(N)$,

$$\mathcal{J}(\theta) \geq \alpha_{\max}\gamma. \tag{15}$$

Proof Let $\theta \in \mathcal{D}_\gamma(N)$. Then $\mathcal{J}(\theta) = \theta^{\mathrm{T}} M\theta = \theta^{\mathrm{T}}(M - \alpha_{\max}N + \alpha_{\max}N)\theta = \theta^{\mathrm{T}}(M - \alpha_{\max}N)\theta + \alpha_{\max}\gamma \geq \alpha_{\max}\gamma$. □

Lemma 5.2 *For all* $\alpha \in \mathcal{S}\backslash\{\alpha_{\max}\}$,

$$\mathcal{N}(M - \alpha N) \cap \mathcal{D}_\gamma(N) = \varnothing. \tag{16}$$

Proof From Proposition 5.1 it follows that, for all $\theta \in \mathcal{D}_\gamma(N)$, $\mathcal{J}(\theta) \geq \alpha_{\max}\gamma$. Next, if $\mathcal{S} = \{0\}$, then (16) need not be verified. Hence assume $\alpha_{\max} > 0$, let $\alpha \in [0\ \alpha_{\max})$, and suppose that $\mathcal{N}(M - \alpha N) \cap \mathcal{D}_\gamma(N) \neq \varnothing$. Then, for all $\hat{\theta} \in \mathcal{N}(M - \alpha N) \cap \mathcal{D}_\gamma(N)$, $\mathcal{J}(\hat{\theta}) = \hat{\theta}^{\mathrm{T}}(M - \alpha N + \alpha N)\hat{\theta} = \hat{\theta}^{\mathrm{T}}(M - \alpha N)\hat{\theta} + \alpha\gamma = \alpha\gamma < \alpha_{\max}\gamma$, which contradicts $\mathcal{J}(\hat{\theta}) \geq \alpha_{\max}\gamma$. Thus, (16) holds for all $\alpha \in \mathcal{S}\backslash\{\alpha_{\max}\}$. □

Corollary 5.3 *Assume* $\alpha_{\max} > 0$. *Then* $\mathcal{N}(M) \cap \mathcal{D}_\gamma(N) = \varnothing$.

Corollary 5.4 *Assume* $N \leq M$. *Then* $\mathcal{N}(M) \cap \mathcal{D}_\gamma(N) = \varnothing$.

Corollary 5.5 *Assume* $N \leq M$. *If* $\mathcal{N}(M - N) \cap \mathcal{D}_\gamma(N) \neq \varnothing$ *then* $\alpha_{\max} = 1$.

The following result considers existence of solutions to (7).

Theorem 5.6 $\theta \in \mathcal{D}_\gamma(N)$ *satisfies*

$$\mathcal{J}(\theta) = \alpha_{\max}\gamma \tag{17}$$

if and only if $\theta \in \mathcal{N}(M - \alpha_{\max}N) \cap \mathcal{D}_\gamma(N)$. *Hence, if*

$$\mathcal{N}(M - \alpha_{\max}N) \cap \mathcal{D}_\gamma(N) \neq \varnothing, \tag{18}$$

then

$$\min_{\theta \in \mathcal{D}_\gamma(N)} \mathcal{J}(\theta) = \alpha_{\max}\gamma \tag{19}$$

and

$$\mathcal{N}(M - \alpha_{\max}N) \cap \mathcal{D}_\gamma(N) \subseteq \mathcal{W}_\gamma(N), \tag{20}$$

and thus (7) has a solution.

Proof From Proposition 5.1, we have that $\mathcal{J}(\theta) \geq \alpha_{\max}\gamma$. Furthermore, let $\theta \in (M - \alpha_{\max}N) \cap \mathcal{D}_\gamma(N)$. Then $\mathcal{J}(\theta) = \theta^{\mathrm{T}}(M - \alpha_{\max}N)\theta + \alpha_{\max}\gamma = \alpha_{\max}\gamma$. Conversely, let $\theta_1 \in \mathcal{D}_\gamma(N)$ satisfy $\mathcal{J}(\theta_1) = \theta_1^T M\theta_1 = \alpha_{\max}\gamma = \alpha_{\max}\theta_1^{\mathrm{T}}N\theta_1$. Then $\theta_1^{\mathrm{T}}(M - \alpha_{\max}N)\theta_1 = 0$. Since $M - \alpha_{\max}N \geq 0$ it follows that, $\theta_1 \in \mathcal{N}(M - \alpha_{\max}N)$. Hence $\theta_1 \in \mathcal{N}(M - \alpha_{\max}N) \cap \mathcal{D}_\gamma(N)$. The last statement is immediate. □

Several corollaries of Theorem 5.6 are discussed in the Appendix.

Theorem 5.6 provides a necessary and sufficient condition for the cost $\mathcal{J}$ to have a minimum value equal to the lower bound $\alpha_{\max}\gamma$ given by Proposition 5.1. However, Theorem 5.6 does not consider the case in which a solution might exist with a cost that is greater than $\alpha_{\max}\gamma$. Theorem 5.8 shows that this case cannot occur. The following lemma is needed.

Lemma 5.7 *Assume that (7) has a solution, let θ_0 solve (7), and define $\beta \triangleq \mathcal{J}(\theta_0)/\gamma$. Then $\theta_0^{\mathrm{T}}(M - \beta N)\theta_0 = 0$.*

Proof $\mathcal{J}(\theta_0) = \theta_0^{\mathrm{T}}M\theta_0 = \theta_0^{\mathrm{T}}(M - \beta N)\theta_0 + \beta\theta_0^{\mathrm{T}}N\theta_0 = \theta_0^{\mathrm{T}}(M - \beta N)\theta_0 + \beta\gamma = \theta_0^{\mathrm{T}}(M - \beta N)\theta_0 + \mathcal{J}(\theta_0)$. Thus $\theta_0^{\mathrm{T}}(M - \beta N)\theta_0 = 0$. □

Theorem 5.8 *Assume that (7) has a solution, and let θ_0 solve (7). Then*

$$\mathcal{J}(\theta_0) = \alpha_{\max}\gamma \tag{21}$$

and

$$\theta_0 \in \mathcal{N}(M - \alpha_{\max}N). \tag{22}$$

Hence,

$$\mathcal{W}_\gamma(N) \subseteq \mathcal{N}(M - \alpha_{\max}N) \cap \mathcal{D}_\gamma(N). \tag{23}$$

Proof Suppose that $\mathcal{J}(\theta_0) = \beta\gamma$, where $\beta > \alpha_{\max}$. Since $\beta \notin \mathcal{S}$, it follows that there exists $\theta \in \mathbb{R}^n$ such that $\theta^{\mathrm{T}}(M - \beta N)\theta < 0$ and thus $0 \leq \theta^{\mathrm{T}}M\theta < \beta\theta^{\mathrm{T}}N\theta$. Thus $\theta^{\mathrm{T}}N\theta > 0$ and hence $\hat{\theta} \triangleq \sqrt{\frac{\gamma}{\theta^{\mathrm{T}}N\theta}}\theta \in \mathcal{D}_\gamma(N)$. Furthermore, $\hat{\theta}^{\mathrm{T}}(M - \beta N)\hat{\theta} < 0$. Hence $\mathcal{J}(\hat{\theta}) = \hat{\theta}^{\mathrm{T}}M\hat{\theta}^{\mathrm{T}} < \beta\hat{\theta}^{\mathrm{T}}N\hat{\theta} = \beta\gamma = \mathcal{J}(\theta_0)$, which contradicts the assumption that θ_0 solves (7). Thus $\beta \leq \alpha_{\max}$. However, Theorem 5.6 implies that $\mathcal{J}(\theta_0)/\gamma \geq \alpha_{\max}$. Hence $\beta \geq \alpha_{\max}$. Consequently, $\beta = \alpha_{\max}$ and thus (21) holds. Next, it follows from Lemma 5.7 with $\beta = \alpha_{\max}$ that $\theta_0^{\mathrm{T}}(M - \alpha_{\max}N)\theta_0 = 0$. Since $M - \alpha_{\max}N \geq 0$ it follows that (22) holds. □

Corollary 5.9 *Assume $\mathcal{N}(M) \cap \mathcal{D}_\gamma(N) \neq \varnothing$. Then*

$$\min_{\theta \in \mathcal{D}_\gamma(N)} \mathcal{J}(\theta) = 0 \tag{24}$$

and $\alpha_{\max} = 0$.

Combining (20) and (23) we have our main result.

Theorem 5.10

$$\mathcal{W}_\gamma(N) = \mathcal{N}(M - \alpha_{\max}N) \cap \mathcal{D}_\gamma(N). \tag{25}$$

For Example 2 in Table 1, $\mathcal{N}(M - \alpha_{\max}N) \cap \mathcal{D}_\gamma(N) = \varnothing$, and thus (7) does not have a solution.

6 Uniqueness of solutions

The following result shows that, if the QCQP problem (7) has a solution and (M, N) is not regular, then the QCQP problem (7) has an infinite number of solutions.

Proposition 6.1 *Assume* $\theta_0 \in \mathcal{W}_\gamma(N)$. *Then*

$$\theta_0 + [\mathcal{N}(M) \cap \mathcal{N}(N)] \subseteq \mathcal{W}_\gamma(N). \tag{26}$$

Proposition 6.2

$$def(M - \alpha_{\max} N) = 1 \tag{27}$$

if and only if (7) has exactly two solutions. In this case, if $\theta \in \mathcal{N}(M - \alpha_{\max} N)$ *is nonzero and such that* $\theta \notin \mathcal{N}(M) \cap \mathcal{N}(N)$, *then*

$$\mathcal{W}_\gamma(N) = \left\{ \sqrt{\frac{\gamma}{\theta^{\mathrm{T}} N \theta}}\theta, \; -\sqrt{\frac{\gamma}{\theta^{\mathrm{T}} N \theta}}\theta \right\}. \tag{28}$$

Proof Since $\theta \in \mathcal{N}(M - \alpha_{\max} N)$, it follows that $\theta^{\mathrm{T}} M \theta = \alpha_{\max} \theta^{\mathrm{T}} N \theta$. If $\theta^{\mathrm{T}} N \theta < 0$, then it follows that $\theta^{\mathrm{T}} M \theta < 0$, which contradicts $M \geq 0$. Next, suppose $\theta^{\mathrm{T}} N \theta = 0$. Then it follows that $\theta^{\mathrm{T}} M \theta = 0$ and thus $\theta \in \mathcal{N}(M)$. Next, since $\alpha_{\max} > 0$ and $\theta \in \mathcal{N}(M - \alpha_{\max} N) \cap \mathcal{N}(M)$, it follows that $\theta \in \mathcal{N}(N)$, which contradicts $\theta \notin \mathcal{N}(M) \cap \mathcal{N}(N)$. Thus $\theta^{\mathrm{T}} N \theta > 0$. Thus it follows that (28) holds. □

Proposition 6.3 *Assume* $\alpha_{\max} > 0$ *and there exists nonzero* $\phi \in \mathbb{R}^n$ *satisfying* $\phi^{\mathrm{T}} M \phi = \phi^{\mathrm{T}} N \phi = 0$. *Then, if (7) has a solution, it has an infinite number of solutions.*

Proof Let θ_1 solve (7) and let $\beta \in \mathbb{R}$. Then it follows from Theorem 5.10 that, $\theta_1 \in \mathcal{N}(M - \alpha_{\max} N) \cap \mathcal{D}_\gamma(N)$. Next, since $\phi \in \mathcal{N}(M)$, it follows that

$$(\theta_1 + \beta\phi)^{\mathrm{T}} M (\theta_1 + \beta\phi) = \theta_1^{\mathrm{T}} M \theta_1.$$

Next, note that $(\theta_1 + \beta\phi)^{\mathrm{T}} N (\theta_1 + \beta\phi) = \theta_1^{\mathrm{T}} N \theta_1 + 2\beta\theta_1^{\mathrm{T}} N \phi + \beta^2 \phi^{\mathrm{T}} N \phi = \theta_1^{\mathrm{T}} N \theta_1 + 2\beta\theta_1^{\mathrm{T}} N \phi$. Now, suppose that $\theta_1^{\mathrm{T}} N \phi \neq 0$. Then, since $\theta_1^{\mathrm{T}} M \theta_1 = \alpha_{\max} \theta_1^{\mathrm{T}} N \theta_1$, it follows that $\theta_1^{\mathrm{T}} M \theta_1 - \alpha_{\max} \theta_1^{\mathrm{T}} N \theta_1 - \alpha_{\max} \theta_1^{\mathrm{T}} N \phi \neq 0$. Hence $\theta_1^{\mathrm{T}} (M\theta_1 - \alpha_{\max} N(\phi + \theta_1)) \neq 0$ and thus, since $M\phi = 0$, $\theta_1^{\mathrm{T}} (M\theta_1 - \alpha_{\max} N(\phi + \theta_1)) + \theta_1^{\mathrm{T}} M \phi \neq 0$. Thus $\theta_1^{\mathrm{T}} (M - \alpha_{\max} N)(\phi + \theta_1) \neq 0$, which contradicts $\theta_1 \in \mathcal{N}(M - \alpha_{\max} N)$. Thus $\theta_1^{\mathrm{T}} N \phi = 0$. Consequently,

$$(\theta_1 + \beta\phi)^{\mathrm{T}} N (\theta_1 + \beta\phi) = \theta_1^{\mathrm{T}} N \theta_1.$$

Hence, $\theta_1 + \beta\phi$ solves (7) for all $\beta \in \mathbb{R}$, and thus (7) has an infinite number of solutions. □

Example 7 in Table 1 shows that the converse of Proposition 6.3 is not true. Since $\alpha_{\max} = 1$ and $\mathcal{N}(M - \alpha_{\max} N) = \mathbb{R}^2$, it follows from Theorem 5.10 that all $\theta \in \mathbb{R}$ satisfying $\theta^{\mathrm{T}} N \theta = \gamma$ are solutions of (7). However, there does not exist a nonzero $\phi \in \mathbb{R}^2$ such that $\phi^{\mathrm{T}} M \phi = \phi^{\mathrm{T}} N \phi = 0$.

7 Positive-semidefinite and positive-definite constraints

7.1 Positive-semidefinite N

The following result relates the sufficient conditions for existence of a solution to (7) given by Corollary 8.2 to the sufficient conditions for existence given in [10] for the quadratic programming problem with $N \geq 0$.

Proposition 7.1 *Assume that $N \geq 0$. If (M, N) is regular then*

$$\min_{\theta \in \mathcal{D}_\gamma(N)} \mathcal{J}(\theta) = \alpha_{\max}\gamma, \tag{29}$$

and thus (7) has a solution.

Proof The result follows from (vii) in Proposition 3.2 and Corollary 8.2. □

We now consider

$$N = \begin{bmatrix} \gamma I_r & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(n-r)} \end{bmatrix}, \tag{30}$$

where $1 \leq r \leq 2n+1$, and partition $M = \begin{bmatrix} M_1 & M_2 \\ M_2^{\mathrm{T}} & M_3 \end{bmatrix}$, where $M_1 \in \mathbb{R}^{r\times r}$.

Proposition 7.2 *Assume that N is given by (30). Then $\alpha_{\max} = 0$ if and only if M_1 is singular.*

Proof Sufficiency follows from $iv)$ of Proposition 4.8. To prove necessity, suppose that $\alpha_{\max} = 0$ and $M_1 > 0$. Then it follows from Proposition 4.4 that $\alpha_{\max} \geq \lambda_{\min}(M_1)/\lambda_{\max}(I_r) = \lambda_{\min}(M_1) > 0$, which contradicts $\alpha_{\max} = 0$. Thus M_1 is singular. □

Proposition 7.3 *Assume that N is given by (30). Then*

$$\mathcal{W}_\gamma(N) = \left\{ \sqrt{\frac{\gamma}{\theta^{\mathrm{T}} N\theta}}\theta : \ \theta \in \mathcal{N}(M - \alpha_{\max}N) \backslash \mathcal{N}(N) \right\}. \tag{31}$$

7.2 Positive-definite N

Proposition 7.4 *Assume that $N > 0$. Then*

$$\mathcal{W}_\gamma(N) = \left\{ \sqrt{\frac{\gamma}{\theta^{\mathrm{T}} N\theta}}\theta : \ 0 \neq \theta \in \mathcal{N}(M - \alpha_{\max}N) \right\}, \tag{32}$$

and

$$\min_{\theta \in \mathcal{D}_\gamma(N)} \mathcal{J}(\theta) = \alpha_{\max}\gamma. \tag{33}$$

If, in addition, M is singular, then $\alpha_{\max} = 0$.

Proof Since $\theta_1^{\mathrm{T}} N\theta > 0$ for all nonzero $\theta_1 \in \mathbb{R}^n$, it follows from Theorem 5.10 that (32) holds and it follows from Corollary 8.6 that (33) holds. Now assume M is singular and let $\theta \in \mathcal{N}(M)$ be nonzero. It then follows that, for all $\alpha > 0$, $\theta^{\mathrm{T}}(M - \alpha N)\theta = -\alpha\theta^{\mathrm{T}} N\theta < 0$. Hence $\mathcal{S} = \{0\}$ and thus $\alpha_{\max} = 0$. □

8 Conclusions

In this paper, we considered a quadratic programming problem with a sign-indefinite quadratic equality constraint. We derived necessary and sufficient conditions for existence of solutions involving the generalized eigenvalue $\alpha_{\max}$ of the matrix pencil involving the positive-semidefinite cost matrix M and the sign-indefinite constraint matrix N. We then provided a complete characterization of the solution set in terms of the eigenspace $\mathcal{N}(M - \alpha_{\max}N)$.

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Appendix

The following seven results are corollaries of Theorem 5.6.

Corollary 8.1 *Assume* $\alpha_{\max} = 0$ *and* $\mathcal{N}(M) \cap \mathcal{D}_\gamma(N) \neq \varnothing$. *Then, for all* $\theta \in \mathcal{N}(M) \cap \mathcal{D}_\gamma(N)$,

$$\mathcal{J}(\theta) = 0 \tag{34}$$

and thus

$$\min_{\theta \in \mathcal{D}_\gamma(N)} \mathcal{J}(\theta) = 0. \tag{35}$$

Hence (7) has a solution.

Corollary 8.2 *Assume there exists* $\beta \in \mathbb{R}$ *such that* $\beta N < M$. *Then*

$$\min_{\theta \in \mathcal{D}_\gamma(N)} \mathcal{J}(\theta) = \alpha_{\max}\gamma, \tag{36}$$

and thus (7) has a solution.

Proof Since $M - \alpha_{\max}N$ is singular, let $\theta \in \mathcal{N}(M - \alpha_{\max}N)$ and note that $\beta\theta^{\mathrm{T}}N\theta < \theta^{\mathrm{T}}M\theta = \alpha_{\max}\theta^{\mathrm{T}}N\theta$. Suppose that $\theta^{\mathrm{T}}N\theta = 0$. Then $0 < \theta^{\mathrm{T}}M\theta = 0$, which is a contradiction. Next, suppose that $\theta^{\mathrm{T}}N\theta < 0$. If $\alpha_{\max} > 0$ then $\theta^{\mathrm{T}}M\theta < 0$, which contradicts $M \geq 0$. If $\alpha_{\max} = 0$, then $\beta\theta^{\mathrm{T}}N\theta < \theta^{\mathrm{T}}M\theta = 0$. Hence $\beta > 0$, which implies $\alpha_{\max} > 0$, which is a contradiction. Hence $\theta^{\mathrm{T}}N\theta > 0$. Consequently, $\hat{\theta} \triangleq \sqrt{\frac{\gamma}{\theta^{\mathrm{T}}N\theta}}\theta \in \mathcal{N}(M - \alpha_{\max}N) \cap \mathcal{D}_\gamma(N) \neq \varnothing$. The result now follows from Theorem 5.6. □

Corollary 8.3 *Assume* $\alpha_{\max} > 0$ *and* $\mathcal{N}(M - \alpha_{\max}N) \cap \mathcal{N}(M) = \{0\}$. *Then*

$$\min_{\theta \in \mathcal{D}_\gamma(N)} \mathcal{J}(\theta) = \alpha_{\max}\gamma, \tag{37}$$

and thus (7) has a solution.

Proof Let $\theta \in \mathcal{N}(M - \alpha_{\max}N)$ be nonzero. Then $\theta^{\mathrm{T}}M\theta = \theta^{\mathrm{T}}(M - \alpha_{\max}N + \alpha_{\max}N)\theta = \alpha_{\max}\theta^{\mathrm{T}}N\theta$. Since $\theta \notin \mathcal{N}(M)$, it follows that $\theta^{\mathrm{T}}M\theta > 0$ and thus, $\theta^{\mathrm{T}}N\theta = \theta^{\mathrm{T}}M\theta/\alpha_{\max} > 0$. Hence, $\hat{\theta} \triangleq \sqrt{\frac{\gamma}{\theta^{\mathrm{T}}N\theta}}\theta \in \mathcal{D}_\gamma(N)$. Finally, since $\hat{\theta} \in \mathcal{N}(M - \alpha_{\max}N)$, it follows that $\hat{\theta} \in \mathcal{N}(M - \alpha_{\max}N) \cap \mathcal{D}_\gamma(N)$. The result now follows from Theorem 5.6. □

Corollary 8.4 *Assume* $M > 0$. *Then*

$$\min_{\theta \in \mathcal{D}_\gamma(N)} \mathcal{J}(\theta) = \alpha_{\max}\gamma, \tag{38}$$

and thus (7) has a solution.

Corollary 8.5 *Assume* $\theta^{\mathrm{T}}M\theta > 0$ *for all nonzero* θ *satisfying* $\theta^{\mathrm{T}}N\theta = 0$. *Then*

$$\min_{\theta \in \mathcal{D}_\gamma(N)} \mathcal{J}(\theta) = \alpha_{\max}\gamma, \tag{39}$$

and thus (7) has a solution.

Proof Let $\theta \in \mathcal{N}(M - \alpha_{\max} N)$. Then it follows that

$$\theta^{\mathrm{T}} M\theta = \alpha_{\max}\theta^{\mathrm{T}} N\theta. \tag{40}$$

First, let $\theta^{\mathrm{T}} M\theta = 0$. Then it follows from (40) that either $\alpha_{\max} = 0$ or $\theta^{\mathrm{T}} N\theta = 0$. If $\theta^{\mathrm{T}} N\theta = 0$, then it follows from assumption that $\theta^{\mathrm{T}} M\theta > 0$, which is a contradiction. Hence $\alpha_{\max} = 0$. It follows from Corollary 4.10 that there exists nonzero θ_1 such that $\theta_1^{\mathrm{T}} M\theta_1 = 0$ and $\theta_1^{\mathrm{T}} N\theta_1 \geq 0$. Again by assumption, $\theta_1^{\mathrm{T}} N\theta_1 \neq 0$, and thus $\theta_1^{\mathrm{T}} N\theta_1 > 0$. Furthermore, since $\theta_1 \in \mathcal{N}(M)$ and $\alpha_{\max} = 0$, it follows that $\theta_1 \in \mathcal{N}(M - \alpha_{\max} N)$ and hence $\hat{\theta} \triangleq \sqrt{\frac{\gamma}{\theta_1^T N\theta_1}}\theta_1 \in \mathcal{N}(M - \alpha_{\max} N) \cap \mathcal{D}_\gamma(N)$. The result now follows from Theorem 5.6.

Next, let $\theta^{\mathrm{T}} M\theta > 0$. Then, since $\alpha_{\max} \geq 0$, it follows from (40) that $\alpha_{\max} > 0$ and $\theta^{\mathrm{T}} N\theta > 0$. Hence $\hat{\theta} \triangleq \sqrt{\frac{\gamma}{\theta^T N\theta}}\theta \in \mathcal{D}_\gamma(N)$. Finally, since $\hat{\theta} \in \mathcal{N}(M - \alpha_{\max} N)$, it follows that $\hat{\theta} \in \mathcal{N}(M - \alpha_{\max} N) \cap \mathcal{D}_\gamma(N)$. The result now follows from Theorem 5.6. □

Since $M \geq 0$, the assumption in Corollary 8.5 is a sufficient condition for $\mathbb{S}$ to be closed [9, Theorem 2].

Corollary 8.6 *Assume $N > 0$. Then*

$$\min_{\theta \in \mathcal{D}_\gamma(N)} \mathcal{J}(\theta) = \alpha_{\max}\gamma, \tag{41}$$

and thus (7) has a solution.

Corollary 8.7 *Assume $\alpha_{\max} > 0$. Then (7) has a solution.*

Proof From Proposition 4.6, it follows that there exists nonzero $\theta \in \mathcal{N}(M - \alpha_{\max} N)$ such that $\theta \notin \mathcal{N}(M) \cap \mathcal{N}(N)$. Suppose $\theta \in \mathcal{N}(M)$, then $\theta \in \mathcal{N}(N)$, which implies that $\theta \in \mathcal{N}(M) \cap \mathcal{N}(N)$, which contradicts $\theta \notin \mathcal{N}(M) \cap \mathcal{N}(N)$. Thus $\theta \notin \mathcal{N}(M)$ and hence $\theta^{\mathrm{T}} M\theta > 0$. Since $\theta^{\mathrm{T}} M\theta = \alpha_{\max}\theta^{\mathrm{T}} N\theta$, it follows that $\theta^{\mathrm{T}} N\theta > 0$ and thus $\sqrt{\frac{\gamma}{\theta^{\mathrm{T}} N\theta}}\theta$ solves (7). □

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